The author thanks V. I. Iudovich for continuing interest and valuable advice.

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Translated by L. K.

THE EULER APPROXIMATION FOR COLLISIONLESS POLYDISPERSE SUSPENSIONS

PMM Vol. 34, №2, 1970, pp. 308-317
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(Received November 4, 1969)

The kinetic equations for various fractions of the dispersed phase of a polydisperse suspension and the system of dynamic equations defining the motion of the suspension as a set of interpenetrating continua are formulated. It is assumed that the suspension is "collisionless", i.e. that its particles interact largely by way of the random velocity and pressure fields in the dispersion medium. The relations characterizing the structure of the random pulsations of the suspension phases ("pseudoturbulence") are considered without allowance for the derivatives of the dynamic variables describing the mean motion. This makes it possible to obtain the dynamic equations in an approximation analogous to the Euler approximation in the hydromechanics of single-phase media. The equations of pseudo-turbulent particle energy transfer which close the system of dynamic equations are written out in the same approximation.

A hydrodynamic model of a polydisperse suspension which adequately describes its mechanical behavior in the continuum approximation can be constructed by a natural generalization of the method already applied to a monodisperse suspension (e.g. see [1]). To avoid repetition, many of the concepts discussed in detail in the case of a monodisperse suspension are used here without further explanation. For clarity we begin with the case where the disperse phase can be represented as a collection of a finite set of fractions. The results thus obtained are then extended to suspensions with continuous particle distributions.

Let us consider the particles constituting N distinct fractions suspended in a fluid of viscosity μ_0 and density d_0. We denote the radius, density, and volume of a particle of the jth fraction by a_j, d_j and σ_j, respectively (the index j here and below denotes the number of the fraction: $j = 1, 2, .., N$). We shall also make use of the specific volume of a particle in the suspension and its "associated" volume, i. e. the fluid-filled part of its specific volume.

We introduce the velocity $\mathbf{w}^{(j)}$ of some particle, the velocity and pressure $\mathbf{v}^{(j)}$ and p_j of the fluid in its associated volume, and the local concentration ρ_j of the suspension defined as the ratio of the volume σ_j to the specific volume of a given particle. Denoting any of the above quantities by φ_j, we can write

$$\varphi_j = \langle\varphi_j\rangle + \varphi_j', \qquad \langle\varphi_j'\rangle \equiv 0 \tag{0.1}$$

where the first term is the value of φ_j averaged over the ensemble and the second term the pseudoturbulent pulsation of φ_j. In addition, we introduce (as in [1]) quantities of the form φ_j° which are the result of averaging φ_j over the nominal distributions characterizing the probability of realization of various values of $\mathbf{v}^{(j)}$, p_j, ρ_j for a fixed particle velocity $\mathbf{w}^{(j)}$.

We have

$$\varphi_j^\circ = \langle\varphi_j\rangle + \varphi_j'', \qquad \langle\varphi_j\rangle \equiv \langle\varphi_j^\circ\rangle_f, \qquad \langle\varphi_j''\rangle_f \equiv 0 \tag{0.2}$$

where $\langle\ \rangle_j$ denotes averaging over the distribution function of the particles of the jth fraction over the velocities f_j.

We assume on the basis of physical considerations that the dynamic variables $\langle\varphi_j\rangle$ pertaining to the motion of the fluid do not depend on the number j. In addition, we assume that the statistical properties of the ratio of σ_j to the specific volume are the same for all j. Then

$$\langle\mathbf{v}^{(j)}\rangle = \langle\mathbf{v}\rangle, \quad \langle p_j\rangle = \langle p\rangle, \quad \rho_j = \rho = \langle\rho\rangle + \rho' \tag{0.3}$$

The second of the above assumptions is generally invalid for various regular particle packings. However, the pulsations of the suspended particles enable us to regard their real packing as the result of averaging over various types of regular packings, so that the assumption appears to be quite plausible. In any case, it is inevitable in the context of the statistical theory based on the notion of chaotic packing of the particles in the suspension at any given instant.

1. The kinetic equations. The kinetic equation for the particles of an arbitrary jth fraction is readily derivable of the method of [1]. We have

$$\frac{D^{(j)}f_j}{Dt} + \mathbf{w}^{(j)'}\frac{\partial f_j}{\partial\mathbf{r}} + \frac{\partial}{\partial\mathbf{w}^{(j)'}}\left(\left[\frac{\mathbf{F}_p^{(j)\circ}}{m_j} - \frac{D^{(j)}\langle\mathbf{w}^{(j)}\rangle}{Dt}\right)f_j\right] - \tag{1.1}$$

$$-\left(\frac{\partial f_j}{\partial\mathbf{w}^{(j)'}}*\mathbf{w}^{(j)'}\right):\left(\frac{\partial}{\partial\mathbf{r}}*\langle\mathbf{w}^{(j)'}\rangle\right) = \frac{1}{m_j}\left(\frac{\partial}{\partial\mathbf{w}^{(j)'}}*\frac{\partial}{\partial\mathbf{w}^{(j)'}}\right):(\mathbf{A}^{(j)}f_j) + \left(\frac{\partial f_j}{\partial t}\right)_c$$

$$\frac{D^{(j)}}{Dt} = \frac{\partial}{\partial t} + \langle\mathbf{w}^{(j)}\rangle\frac{\partial}{\partial\mathbf{r}}, \quad \mathbf{a}*\mathbf{b} = \|a_i b_l\|, \ \mathbf{A}:\mathbf{B} = A_{il}B_{li}, \ m_j = d_j\sigma_j$$

The solution f_j of this equation is some "average" distribution in the sense that averaging over the nominal distributions [1] is assumed to have been carried out. The quantity $\mathbf{F}_p^{(j)\circ}$ is the total force exerted by the fluid stream and external fields on a particle of the jth fraction averaged according to (0.2); $A^{(j)}$ is some unknown tensor describing diffusion in the velocity space [1–3]. The last term in the right side of (1.1) represents the change in f_j due to direct particle collisions.

For simplicity we assume that the role of direct collisions in the transfer of momentum and energy between particles is relatively small, i.e. that particle interaction occurs largely by way of the fluid filling the gaps between particles. Close analysis shows that this assumption is valid for a very broad class of practically important disperse systems, provided their concentration is not too close to the concentration of a densely-packed particle system. The collision term in Eq. (1.1) for such disperse systems (which can be aptly called "collisionless") can be approximated by zero, which simplifies the subsequent computations considerably.

The expression for the forces $\mathbf{F}_p^{(j)}$ can be written as

$$\mathbf{F}_p^{(j)} = m_j\mathbf{g} + \mathbf{F}_i^{(j)}, \quad \langle \mathbf{F}_p^{(j)} \rangle = m_j\mathbf{g} + \langle \mathbf{F}_i^{(j)} \rangle, \quad \mathbf{F}_p^{(j)'} = \mathbf{F}_i^{(j)'} \tag{1.2}$$

where $\mathbf{g}$ is the acceleration of the external mass field. The "dissipative" component of force (1.2) has been omitted by virtue of our neglect of direct collisions [1]. The force of interaction of the particles with the ambient fluid $\mathbf{F}_i^{(j)}$ can be written as

$$\mathbf{F}_i^{(j)} = -\sigma_j \frac{\partial p_j}{\partial \mathbf{r}} + \varkappa_j m_j \Big[\beta_{1j} K_1(\rho)\,\mathbf{u}^{(j)} + \beta_{2j} K_2(\rho)\, u^{(j)}\mathbf{u}^{(j)} + \xi(\rho)\frac{d^{(j)}\mathbf{u}^{(j)}}{dt} +$$

$$+ \gamma_j \int_{-\infty}^{t} \eta(\rho) \frac{d^{(j)}\mathbf{u}^{(j)}}{dt}\bigg|_{t=t'} \frac{dt'}{\sqrt{t-t'}}\Big], \quad \varkappa_j = \frac{d_0}{d_j}, \quad \mathbf{u}^{(j)} = \mathbf{v}^{(j)} - \mathbf{w}^{(j)} \tag{1.3}$$

with the aid of (0.2).

Here β_{1j}, β_{2j}, γ_j are some coefficients which depend on the physical properties of the jth fraction and of the fluid; K_i, ξ, η are functions which allow for the restriction of flow past particles in the system. Differentiation over time in (1.3) is carried out along the particle trajectory. Experimental values of K_i are given in [4] and elsewhere; the specific form of these functions and of the coefficients β_{ij}, γ_j has no direct bearing on our study. We note that expression (1.3) differs from the analogous expression of [1] only by a term quadratic in the relative velocity $\mathbf{u}^{(j)}$.

Making use of relations (0.1), (0.3), we obtain from (1.3) the following expressions [1] valid to within second-order terms in the pseudoturbulent variables:

$$\langle \mathbf{F}_i^{(j)} \rangle \approx -\sigma_j \frac{\partial \langle p \rangle}{\partial \mathbf{r}} + \varkappa_j m_j \Big\{\beta_{1j}\Big[K_1 \langle \mathbf{u}^{(j)} \rangle + \frac{dK_1}{d\langle\rho\rangle}\langle \rho' \mathbf{u}^{(j)'} \rangle +$$

$$+ \frac{1}{2}\frac{d^2K_1}{d\langle\rho\rangle^2}\langle \mathbf{u}^{(j)} \rangle \langle \rho'^2 \rangle\Big] + \beta_{2j}\Big[K_2\big(\langle u^{(j)}\rangle\langle \mathbf{u}^{(j)} \rangle + \langle (\mathbf{u}_0^{(j)}\mathbf{u}^{(j)'})\,\mathbf{u}^{(j)'} \rangle +$$

$$+ \frac{1}{2}\langle u^{(j)'2} - (\mathbf{u}_0^{(j)}\mathbf{u}^{(j)'})^2 \rangle \mathbf{u}_0^{(j)}\big) + \frac{dK_2}{d\langle\rho\rangle}\big(\langle u^{(j)}\rangle \langle \rho' \mathbf{u}^{(j)'} \rangle +$$

$$+ \langle \mathbf{u}^{(j)} \rangle \langle \rho' (\mathbf{u}_0^{(j)}\mathbf{u}^{(j)'}) \rangle\big) + \frac{1}{2}\frac{d^2K_2}{d\langle\rho\rangle^2}\langle u^{(j)}\rangle\langle \mathbf{u}^{(j)} \rangle \langle \rho'^2 \rangle\Big] +$$

$$+ \xi \frac{D^{(j)}\langle \mathbf{u}^{(j)} \rangle}{Dt} + \frac{d\xi}{d\langle\rho\rangle}\Big\langle \rho'\Big(\frac{d^{(j)}\mathbf{u}^{(j)'}}{dt} + \Big(\mathbf{w}^{(j)'}\frac{\partial}{\partial \mathbf{r}}\Big)\langle \mathbf{u}^{(j)} \rangle\Big)\Big\rangle + \tag{1.4}$$

$$+ \frac{1}{2}\frac{d^2\xi}{d\langle\rho\rangle^2}\frac{D^{(j)}\langle \mathbf{u}^{(j)} \rangle}{Dt}\langle \rho'^2 \rangle + \gamma_j \int_{-\infty}^{t}\Big[\eta\frac{D^{(j)}\langle \mathbf{u}^{(j)} \rangle}{Dt} + \frac{d\eta}{d\langle\rho\rangle}\Big\langle \rho'\Big(\frac{d^{(j)}\mathbf{u}^{(j)'}}{dt} +$$

$$+ \Big(\mathbf{w}^{(j)'}\frac{\partial}{\partial \mathbf{r}}\Big)\langle \mathbf{u}^{(j)} \rangle\Big)\Big\rangle + \frac{1}{2}\frac{d^2\eta}{d\langle\rho\rangle^2}\frac{D^{(j)}\langle \mathbf{u}^{(j)} \rangle}{Dt}\langle \rho'^2 \rangle\Big]_{t=t'}\frac{dt'}{\sqrt{t-t'}}\Big\}$$

$$\mathbf{F}_i^{(j)'} \approx -\sigma_j \frac{\partial p_j'}{\partial \mathbf{r}} + \varkappa_j m_j \Big\{ \beta_{1j} \Big(K_1 \mathbf{u}^{(j)'} + \frac{dK_1}{d\langle\rho\rangle} \langle \mathbf{u}^{(j)} \rangle \rho' \Big) + \tag{1.5}$$

$$+ \beta_{2j} \Big[K_2 (\langle u^{(j)} \rangle \mathbf{u}^{(j)'} + (\mathbf{u}_0^{(j)} \mathbf{u}^{(j)'}) \langle \mathbf{u}^{(j)} \rangle) + \frac{dK_2}{d\langle\rho\rangle} \langle u^{(j)} \rangle \langle \mathbf{u}^{(j)} \rangle \rho' \Big] +$$

$$+ \xi \Big(\frac{d^{(j)} \mathbf{u}^{(j)'}}{dt} + \Big(\mathbf{w}^{(j)'} \frac{\partial}{\partial \mathbf{r}} \Big) \langle \mathbf{u}^{(j)} \rangle \Big) + \frac{d\xi}{d\langle\rho\rangle} \frac{D^{(j)} \langle \mathbf{u}^{(j)} \rangle}{Dt} \rho' +$$

$$+ \gamma_j \int_{-\infty}^{t} \Big[\eta \Big(\frac{d^{(j)} \mathbf{u}^{(j)'}}{dt} + \Big(\mathbf{w}^{(j)'} \frac{\partial}{\partial \mathbf{r}} \Big) \langle \mathbf{u}^{(j)} \rangle \Big) \Big]_{t=t'} \frac{dt'}{\sqrt{t-t'}}$$

$$\mathbf{u}_0^{(j)} = \langle \mathbf{u}^{(j)} \rangle / \langle u^{(j)} \rangle, \quad K_i \equiv K_i(\langle\rho\rangle), \quad \xi \equiv \xi(\langle\rho\rangle), \quad \eta \equiv \eta(\langle\rho\rangle)$$

The expression for the force $\mathbf{F}_p^{(j)\circ}$ occurring in (1.1) can be obtained as in [1]. We begin by expressing the quantity φ_j'' in (0.2) in the form $\varphi_j'' = \mathbf{s}^{(j)}[\varphi]\,\mathbf{w}^{(j)'}$, where $\mathbf{s}^{(j)}[\varphi]$ is some unknown tensor quantity. In the case of the pseudoturbulent pulsations occurring in the right side of relation (1.5) we have

$$\rho'' = \mathbf{s}^{(j)}[\rho]\,\mathbf{w}^{(j)'}, \qquad \mathbf{u}^{(j)''} = \mathbf{s}^{(j)}[\mathbf{u}]\,\mathbf{w}^{(j)'}$$

$$-\nabla p^{(j)''} = \mathbf{s}^{(j)}[-\nabla p]\,\mathbf{w}^{(j)'}, \qquad \frac{d_j \mathbf{u}^{(j)''}}{dt} = \mathbf{s}^{(j)} \Big[\frac{d\mathbf{u}}{dt} \Big] \mathbf{w}^{(j)'} \tag{1.6}$$

The force $\mathbf{F}_p^{(j)\circ}$ can therefore be expressed as the sum of $\langle \mathbf{F}_p^{(j)} \rangle$ as given by (1.2), (1.4) and the quantity $\mathbf{F}_p^{(j)'}$ as given by (1.5), where all of the pseudoturbulent variables have been replaced by linear functions of $\mathbf{w}^{(j)'}$ in accordance with formulas (1.6).

2. The mass and momentum conservation equations. The dynamic equations for the jth particle fraction considered in the continuum approximation are readily derivable from (1.1) by means of standard operations. As in [1] we obtain the equations

$$\frac{D^{(j)} \langle\rho\rangle_j}{Dt} + \langle\rho\rangle_j \frac{\partial \langle \mathbf{w}^{(j)} \rangle}{\partial \mathbf{r}} = 0, \qquad \sum_{j=1}^{N} \langle\rho\rangle_j \equiv \langle\rho\rangle$$

$$d_j \langle\rho\rangle_j \frac{D^{(j)} \langle \mathbf{w}^{(j)} \rangle}{Dt} = -\frac{\partial \mathbf{P}^{(p)(j)}}{\partial \mathbf{r}} + \frac{\langle\rho\rangle_j}{\sigma_j} \langle \mathbf{F}_p^{(j)} \rangle \tag{2.1}$$

$$\mathbf{P}^{(p)(j)} = d_j \langle\rho\rangle_j \langle \mathbf{w}^{(j)'} * \mathbf{w}^{(j)'} \rangle$$

The quantity $\langle\rho\rangle_j$ is the average volume particle concentration of the jth fraction in the system (not to be confused with the ρ_j in (0.3) which characterizes the concentration of the suspension near the particle).

The dynamic equations for the fluid phase are also obtainable by our old method [1] from the Navier-Stokes equations for the motion of the fluid in the gaps between particles. Making use of relations (0.3), we obtain the equations

$$\Big(\frac{\partial}{\partial t} + \langle \mathbf{v} \rangle \frac{\partial}{\partial \mathbf{r}} \Big) \langle\rho\rangle - (1 - \langle\rho\rangle) \frac{\partial \langle \mathbf{v} \rangle}{\partial \mathbf{r}} - \frac{\partial \mathbf{q}}{\partial \mathbf{r}} = 0, \quad \mathbf{q} = -\sum_{j=1}^{N} \frac{\langle\rho\rangle_j}{\langle\rho\rangle} \langle \rho' \mathbf{v}^{(j)'} \rangle$$

$$d_0 \Big[\frac{\partial}{\partial t} ((1 - \langle\rho\rangle) \langle \mathbf{v} \rangle) + \frac{\partial}{\partial \mathbf{r}} ((1 - \langle\rho\rangle) \langle \mathbf{v} \rangle * \langle \mathbf{v} \rangle) + \frac{\partial \mathbf{q}}{\partial t} \Big] = \tag{2.2}$$

$$= -\frac{\partial \mathbf{P}^{(f)}}{\partial \mathbf{r}} - \frac{\partial \langle p \rangle}{\partial r} + \mu_0 \frac{\partial}{\partial \mathbf{r}} \Big(S \langle \mathbf{e} \rangle + \frac{dS}{d\langle\rho\rangle} \sum_{j=1}^{N} \frac{\langle\rho\rangle_j}{\langle\rho\rangle} \langle \rho' \mathbf{e}^{(j)'} \rangle +$$

$$+\frac{1}{2}\frac{d^2S}{d\langle\rho\rangle^2}\langle\mathbf{e}\rangle\langle\rho'^2\rangle\Big)+d_0(1-\langle\rho\rangle)\mathbf{g}-\sum_{j=1}^{N}\frac{\langle\rho\rangle_j}{\sigma_j}\langle\mathbf{F}_i^{(j)}\rangle,\quad S\equiv S(\langle\rho\rangle) \qquad \text{(cont.)}$$

$$\mathbf{P}^{(f)}=d_0\Big[(1-\langle\rho\rangle)\sum_{j=1}^{N}\frac{\langle\rho\rangle_j}{\langle\rho\rangle}\langle\mathbf{v}^{(j)\prime}*\mathbf{v}^{(j)\prime}\rangle+\mathbf{q}*\langle\mathbf{v}\rangle+\langle\mathbf{v}\rangle*\mathbf{q}\Big]$$

Here $\mu = \mu_0 S(\rho)$ is the effective viscosity of the fluid filtered through the particle lattice.

It is clear that as in the theory of monodisperse suspensions we must express all of the pseudoturbulence characteristics occurring in Eqs. (2.1) and (2.2) in terms of the dynamic variables defining the average motion of the suspension and the physical parameters of the two phases.

3. The pseudoturbulent structure of the suspension. Random pseudoturbulent variables satisfy stochastic equations obtainable from the equations of particle motion and the Navier-Stokes equations for the fluid in the same way as the corresponding equations of [1]. Neglecting the ratios of the pseudoturbulence scales to the mean-motion scales, we can write these equations in the form

$$m_j\frac{d^{(j)}\mathbf{w}^{(j)\prime}}{dt}=\mathbf{F}_i^{(j)\prime},\qquad \left(\frac{d^{(j)}}{dt}+\langle\mathbf{u}^{(j)}\rangle\frac{\partial}{\partial\mathbf{r}}\right)\rho'-(1-\langle\rho\rangle)\frac{\partial\mathbf{v}^{(j)\prime}}{\partial\mathbf{r}}=0$$

$$d_0(1-\langle\rho\rangle)\left(\frac{d^{(j)}}{dt}+\langle\mathbf{u}^{(j)}\rangle\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{v}^{(j)\prime}=-\frac{\partial p_j'}{\partial\mathbf{r}}+\mu_0 S\frac{\partial\mathbf{e}^{(j)\prime}}{\partial\mathbf{r}}-\sum_{j=1}^{N}\frac{\langle\rho\rangle_j}{\sigma_j}\mathbf{F}_i^{(j)\prime} \qquad (3.1)$$

As in [1], we have made use of a coordinate system attached to an isolated particle.

It is convenient to express all of the random pseudoturbulent functions in the form of Fourier-Stieltjes integrals. System (3.1) then becomes a system of linear algebraic equations for the spectral measures of these random functions which makes it possible to express all the spectral measures in terms of the single spectral measure $dZ_\rho^{(j)}$ of the random process ρ'. It is then easy to determine all of the spectral densities characterizing the wave structure of the pseudoturbulence in the form

$$\Psi_{\varphi,\psi}^{(j)}(\omega,\mathbf{k})=L_{\varphi,\psi}^{(j)}(\omega,\mathbf{k})\,\Psi_{\rho,\rho}^{(j)}(\omega,\mathbf{k}) \qquad (3.2)$$

$$\Psi_{\varphi,\psi}^{(j)}(\omega,\mathbf{k})\,d\omega d\mathbf{k}\equiv\langle dZ_\varphi^{(j)*}\,dZ_\psi^{(j)}\rangle$$

where $\varphi^{(j)\prime}$, $\psi^{(j)\prime}$ are arbitrary pseudoturbulent variables, $dZ_\varphi^{(j)}$ and $dZ_\psi^{(j)}$ are their spectral measures, and $L_{\varphi,\psi}^{(j)}$ are some known functions of the frequency ω and wave vector $\mathbf{k}$ of the pseudoturbulent pulsations, as well as of various dynamic derivatives and physical parameters. The expressions for $L_{\varphi,\psi}^{(j)}$ obtainable in elementary fashion from the aforementioned system of algebraic equations will not be written out here.

In the case of a monodisperse particle suspension the expression for $\Psi_{\rho\rho}^{(j)}$ is easy to obtain with the aid of the generalized diffusion equation formulated in [5]. With a polydisperse suspension the equation is replaced by N such equations for each of the fractions. It is easy to show that the mean-square concentration fluctuation $\langle\rho'^2\rangle$ of such a suspension consists additively of the quantities $\langle\rho'^2\rangle_j$ characterizing the concentration fluctuations of the various fractions; the dynamic behavior of each such quantity is defined by the corresponding generalized diffusion equation.

Recalling that the velocities $\langle \mathbf{w}^{(j)} \rangle$ are generally different for different j and omitting the intervening operations, we obtain (as in [1, 5]) an expression for the spectral density ρ' of the process,

$$\Psi'^{(j)}_{\rho,\rho}(\omega, \mathbf{k}) \approx \sum_{l=1}^{N} \Big[\frac{\Phi^{(l)}_{\rho,\rho}(\mathbf{k})}{M^{(j)}_l(\omega,\mathbf{k})} \Big(\int \frac{d\omega}{M^{(j)}_l(\omega,\mathbf{k})} \Big)^{-1} \Big]$$

$$M^{(j)}_l(\omega, \mathbf{k}) \approx [\omega - (\langle \mathbf{w}^{(l)} \rangle - \langle \mathbf{w}^{(j)} \rangle)\,\mathbf{k}]^2 + [\mathbf{D}^{(l)}\mathbf{kk} - \omega^2\theta^{(l)-1}\,\mathrm{tr}\,\mathbf{D}^{(l)}]^2 \quad (3.3)$$

$$\Phi^{(j)}_{\rho,\rho}(\mathbf{k}) = \frac{\langle \rho \rangle_j}{\langle \rho \rangle} \Phi_{\rho,\rho}(\mathbf{k}, k_{0j}), \quad \theta^{(j)} = \langle w^{(j)\prime 2} \rangle, \quad \mathrm{tr}\,\mathbf{D}^{(j)} = D^{(j)}_{ii}$$

Here $\mathbf{D}^{(j)}_{ii}$ is the tensor of effective pseudoturbulent diffusion of the particles of the jth fraction, and the quantities $\Phi_{\rho,\rho}$, k_{0j} are defined in [1, 5]. Equations (3.3) close the system of relations (3.2). We note that Eqs. (3.1) and (3.3) were written out under the assumption that the derivatives of the dynamic variables are negligible compared with the corresponding derivatives of the pseudoturbulent variables; this corresponds to the zeroth approximation with respect to the ratios of the pseudoturbulence and mean-motion scales. We can therefore hope to determine dynamic equations (2.1), (2.2) with the same degree of accuracy.

The pseudoturbulence characteristics of second order in the pseudoturbulent variables, as well as the various correlation functions, are readily derivable from (3.2) and (3.3) by ordinary integration over the frequencies ω and wave space $\mathbf{k}$. Precisely such characteristics are employed in the dynamic equations of [1].

We note, however, that the pseudoturbulent quantities computed from (3.2) with the aid of (3.3) and the formula for $\Phi_{\rho,\rho}$ given in [1, 5] apply, strictly speaking, only to states in which the derivatives of all the dynamic variables (the average velocities of the various particle fractions and fluid, the average concentration of the suspension, and the mean pressure gradient) are equal exactly to zero (by analogy with the kinetic theory, we call them "equilibrium" states).

This is because the formulas for $\Phi^{(j)}_{\rho,\rho}$ of (3.3) and for $\Phi_{\rho,\rho}$ of [1, 5] are strictly valid for such states only. The equilibrium pseudoturbulence characteristics computed on the basis of the above results will be identified by a zero subscript in order to avoid confusion. We note that any equilibrium pseudoturbulent characteristic $\langle \varphi^{(j)\prime}\, \psi^{(j)\prime} \rangle_0$ can be expressed as some known function of the dynamic variables and physical parameters by integrating relations (3.2) over ω and $\mathbf{k}$ and then expressing all of the pseudoturbulent quantities occurring in definitions (1.3) according to the rules discussed in [1, 5].

The state of a suspension can differ markedly from the corresponding equilibrium state in real flows. The method of determining the quantities $\langle \varphi^{(j)\prime} \psi^{(j)\prime} \rangle$ from known $\langle \varphi^{(j)\prime}\, \psi^{(j)\prime} \rangle_0$ is discussed in [5] on the basis of the assumption that the equilibrium formula for $\Phi_{\rho,\rho}$ is also applicable under nonequilibrium conditions. In this case the equations for various pseudoturbulent characteristics are obtainable (in principle) by the same method as the equations for the correlation functions in the theory of turbulence. For obvious reasons this model cannot be considered particularly successful. Let us now consider a much simpler model.

Specifically, we introduce the scalars $R^{(j)}\ [\varphi, \psi]$, the vectors $\mathbf{R}^{(j)}\ [\varphi, \psi]$ and the tensors $\mathbf{R}^{(j)}\ [\varphi, \psi]$ such that in the equilibrium state we have

$$\langle \varphi^{(j)'} \psi^{(j)'} \rangle_0 = R^{(j)}[\varphi, \psi] \operatorname{tr} \langle \mathbf{w}^{(j)'} * \mathbf{w}^{(j)'} \rangle_0 = R^{(j)}[\varphi, \psi]\, \theta_0^{(j)}$$

$$\langle \varphi^{(j)} \boldsymbol{\psi}^{(j)'} \rangle_0 = \mathbf{R}^{(j)}[\varphi, \boldsymbol{\psi}] \langle \mathbf{w}^{(j)'} * \mathbf{w}^{(j)'} \rangle_0 \tag{3.4}$$

$$\langle \boldsymbol{\varphi}^{(j)'} * \boldsymbol{\psi}^{(j)'} \rangle_0 = \mathbf{R}^{(j)}[\boldsymbol{\varphi}, \boldsymbol{\psi}] \langle \mathbf{w}^{(j)'} * \mathbf{w}^{(j)'} \rangle_0$$

where the primed quantities in the left sides are arbitrary scalar or vector pseudoturbulent variables. It is clear that the quantities just introduced can be regarded as some known functions of the dynamic variables and physical parameters.

It is possible in principle to solve kinetic equation (1.1) for each fraction in a nonequilibrium state and then to obtain the pseudoturbulent means

$$\langle \mathbf{w}^{(j)'} * \mathbf{w}^{(j)'} \rangle = \frac{1}{n_j} \int (\mathbf{w}' * \mathbf{w}') f_j \, d\mathbf{w}' \tag{3.5}$$

Let us assume that all the other means in the nonequilibrium state have been expressed in terms of $\langle \mathbf{w}^{(j)'} * \mathbf{w}^{(j)'} \rangle$ by relations similar to (3.4). We then say that the quantities $R^{(j)}$ occurring in such relations coincide with the $R^{(j)}$ of (3.4). In fact, (3.4) is valid provided the system is in a state of local equilibrium, i.e. in a state of equilibrium on the level of an individual particle and its immediate neighbors. However, all of the equations considered in the present paper are in fact obtainable by averaging over a time interval Δt satisfying the inequalities $\tau \ll \Delta t \ll T$, where τ and T are the internal and external pseudoturbulence scales [1, 5]. The time τ is essentially the characteristic time required for the establishment of local equilibrium in the system. This means that all of the nonequilibrium states under investigation in the present paper have the property of local equilibrium. We note that the notion of local equilibrium as used here has the same meaning as in nonequilibrium statistical mechanics in general. For example, in the kinetic theory of gases the state of local equilibrium corresponds simply to molecular chaos.

4. The equations of pseudoturbulent particle energy transfer. The equations for the quantities $\theta_i^{(j)} = \langle w_j^{(j)} \rangle^{1/2}$ and their sums over i for particles of different fractions are obtainable in the same way as dynamic equations (2.1). However, in contrast to the dynamic equations these equations contain terms which depend on the unknowns $\mathbf{F}_p^{(j)\circ}$ and $\mathbf{A}^{(j)}$ occurring in kinetic equations (1.1). We shall determine the latter quantities by comparing the results obtained in Sect. 3 with certain results which follow from Eqs. (1.1) in the equilibrium state.

From relations of the form (3.4) written out in the nonequilibrium state we readily obtain the system of linear algebraic equations

$$\langle \boldsymbol{\varphi}^{(j)'} * \mathbf{w}^{(j)'} \rangle = \mathbf{s}^{(j)}[\boldsymbol{\varphi}] \langle \mathbf{w}^{(j)'} * \mathbf{w}^{(j)'} \rangle, \quad \mathbf{s}^{(j)}[\boldsymbol{\varphi}] \equiv \mathbf{R}^{(j)}[\boldsymbol{\varphi}, \mathbf{w}] \tag{4.1}$$

This allows us to assume that the quantities $\mathbf{s}^{(j)}[\boldsymbol{\varphi}]$ are also known, and thus to express the force $\mathbf{F}_p^{(j)\circ}$ solely in terms of the dynamic variables and physical parameters according to the method described at the end of Sect. 1.

We can determine the components of the tensor $\mathbf{A}^{(j)}$ by the method proposed in [1]. Let us consider kinetic equation (1.1) written for the equilibrium state when the derivatives of the dynamic variables (and therefore the derivatives of f_j which depends implicitly on t and $\mathbf{r}$) are identically equal to zero,

$$\frac{\partial}{\partial \mathbf{w}^{(j)'}} [(\mathbf{F}_p^{(j)\circ}) f_j] = \left(\frac{\partial}{\partial \mathbf{w}^{(j)'}} * \frac{\partial}{\partial \mathbf{w}^{(j)'}} \right) : (\mathbf{A}^{(j)} f_j) \tag{4.2}$$

The tensor $\mathbf{A}^{(j)}$ is symmetric by definition, so that we can solve Eq. (4.2) in the principal axes of the tensor $\mathbf{A}^{(j)}$ without limiting generality. Denoting the corresponding eigenvalues of the tensor $\mathbf{A}^{(j)}$ by $A_i^{(j)}$, we seek the solution of (4.2) in the "quasi-Maxwellian" form

$$f_j = n_j \left(\frac{B_1^{(j)} B_2^{(j)} B_3^{(j)}}{\pi^3} \right)^{1/2} \exp\left(-\sum_{i=1}^{3} B_i^{(j)} w_i^{(j)\prime 2} \right) \tag{4.3}$$

where n_j is the countable concentration of particles of the jth fraction. Furthermore, the general expression for the force $\mathbf{F}_p^{(j)\circ}$ is of the form

$$F_{p,i}^{(j)\circ} = G_i^{(j)} - c_i^{(j)} w_i^{(j)\prime} \tag{4.4}$$

(without summation over i), where $\mathbf{G}^{(j)}$ does not depend on $\mathbf{w}^{(j)\prime}$ and where $c_i^{(j)}$ are known functions. Substituting (4.3) and (4.4) into Eq. (4.2) and separating out terms containing differing powers of $\mathbf{w}^{(j)\prime}$ we obtain the following equations:

$$\mathbf{G}^{(j)} = 0, \qquad B_i^{(j)} = \frac{c_i^{(j)}}{2A_i^{(j)}} \tag{4.5}$$

The first of these equations clearly coincides with the equation of conservation of momentum of the jth fraction written out for the equilibrium state; the second equation enables us to express the components of the tensor $A_i^{(j)}$ in terms of the components of the tensor $B_i^{(j)}$ in expression (4.3). The function f_j in (4.3) enables us to compute the mean-square values of the pseudoturbulent particle velocities in various directions. Making use of the second equation of (4.5), we obtain the following equations for the unknowns $A_i^{(j)}$:

$$A_i^{(j)} = c_i^{(j)} \langle w_i^{(j)\prime 2} \rangle_0, \qquad \operatorname{tr} \mathbf{A}^{(j)} = c^{(j)} \theta_0^{(j)} \tag{4.6}$$

The right sides of these equations contain functions of the dynamic variables and physical parameters which can be computed in accordance with the results of Sect. 3.

The distribution functions for the nonequilibrium state in the zeroth approximation in the derivatives of the dynamic variables can also be found in the "quasi-Maxwellian" form

$$f_j = n_j \left(\frac{1}{8\pi^3 \theta_1^{(j)} \theta_2^{(j)} \theta_3^{(j)}} \right)^{1/2} \exp\left(-\sum_{i=1}^{3} \frac{w_i^{(j)\prime 2}}{2\theta_i^{(j)}} \right) \tag{4.7}$$

Making use of relations (4.3), (4.4), (4.6) and (4.7), we obtain from Eqs. (1.1) the following transfer equations for the quantities $\theta_i^{(j)}$ which are the mean squares of the pulsation velocities of the particles of the jth fraction in the ith direction:

$$\frac{D^{(j)} (\langle \rho \rangle_j \theta_i^{(j)})}{Dt} + \langle \rho \rangle_j \theta_i^{(j)} \frac{\partial \langle \mathbf{w}^{(j)} \rangle}{\partial \mathbf{r}} + 2P_{ik}^{(p)(j)} \frac{\partial \langle w_i^{(j)} \rangle}{\partial x_k} = 2c_i^{(j)} \langle \rho \rangle_j (\theta_{i0}^{(j)} - \theta_i^{(j)}) \tag{4.8}$$

Summing (4.8) over i, we also obtain the transfer equation for $\theta^{(j)}$,

$$\frac{D^{(j)} (\langle \rho \rangle_j \theta^{(j)})}{Dt} + \langle \rho \rangle_j \theta^{(j)} \frac{\partial \langle \mathbf{w}^{(j)} \rangle}{D\mathbf{r}} + 2\mathbf{P}^{(p)(j)} : \left(\frac{\partial}{\partial \mathbf{r}} * \langle \mathbf{w}^{(j)} \rangle \right) = 2 \sum_{i=1}^{3} c_i^{(j)} \langle \rho \rangle_j (\theta_{i0}^{(j)} - \theta_i^{(j)}) \tag{4.9}$$

This equation has exactly the same meaning as the heat conduction equation in gas hydrodynamics. Equations (4.8) clearly close the system of dynamic equations (2.1), (2.2). In fact, all of the pseudoturbulent characteristics occurring in the latter equations can be expressed (in accordance with (3.4), (3.5), (4.7)) in terms of the dynamic variables and the quantities $\theta_i^{(j)}$. We therefore have $7N + 4$ equations for determining $7N + 4$ unknowns, i.e. the three components of the velocity $\langle \mathbf{v} \rangle$, the pressure $\langle p \rangle$, $3N$ quantities $\theta_i^{(j)}$, $3N$ velocities $\langle w_i^{(j)} \rangle$, and N mean concentrations of the various fractions $\langle \rho \rangle_j$.

The significance of the above approximation is analogous to that of the Euler approximation in the kinetic theory of gasses and single-phase hydrodynamics. It is apt to retain the same terminilogy for the disperse systems under consideration here.

5. Suspensions with a continuous particle distribution. In order to pass from suspensions characterized by a discrete set of different fractions to suspensions with a continuous particle distribution over some parameter λ (or over several parameters denoted nominally by λ) , we introduce the particle distribution function $\psi(\lambda)$ normalized to the total volume concentration $\langle\rho\rangle$ of the suspension, i.e.

$$\int \psi(\lambda)\, d\lambda = \langle\rho\rangle \tag{5.1}$$

where integration must be carried out over the entire range of variation of λ. The correspondence between the continuous and the discrete description considered above can be established with the aid of the relation

$$\langle\rho\rangle_j = \psi(\lambda_j)\, \Delta\lambda \tag{5.2}$$

where we assume that the coefficients β_{1j}, β_{2j}, γ_j as well as all the other quantities previously accompanied by the index j are some functions of λ. The continuous desription is thus obtainable from the discrete one by taking the limit $\Delta\lambda \to 0$, where λ plays the role of a parameter (or parameters) in the resulting relations.

With allowance for these changes the stochastic equations for $\mathbf{v}'$, $\mathbf{w}'$, p' and ρ' retain their form (3.1), while instead of (3.3) we have

$$\Psi_{\rho,\rho}(\omega, \mathbf{k}) \approx \frac{\Phi_{\rho,\rho}(\mathbf{k})}{\langle\rho\rangle} \int \left[\frac{\psi(\lambda')}{M(\lambda,\lambda')} \left(\int \frac{d\omega}{M(\lambda,\lambda')} \right)^{-1} \right] d\lambda'$$

$$M(\lambda, \lambda') \approx [\omega - \mathbf{k}(\langle \mathbf{w}\rangle(\lambda') - \langle \mathbf{w}\rangle(\lambda))]^2 + [\mathbf{D}(\lambda')\,\mathbf{kk} - \omega^2\theta^{-1}(\lambda')\operatorname{tr}\mathbf{D}(\lambda')]^2 \tag{5.3}$$

where $\langle\rho\rangle$ is expressed in terms of $\psi(\lambda)$ in accordance with (5.1). Making use of (5.3), we can readily compute all the equilibrium pseudoturbulence characteristics in the old way. In this case they also depend on the unknown function $\psi(\lambda)$.

The dynamic equations for the dispersed phase become

$$\frac{D\psi}{Dt} + \psi \frac{\partial \langle \mathbf{w}\rangle}{\partial \mathbf{r}} = 0, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \langle \mathbf{w}\rangle \frac{\partial}{\partial \mathbf{r}}, \quad d = d(\lambda), \ldots \tag{5.4}$$

$$d\psi \frac{D\langle \mathbf{w}\rangle}{Dt} = -\frac{\partial \mathbf{P}^{(p)}}{\partial \mathbf{r}} + \frac{\psi}{\sigma}\langle \mathbf{F}_p\rangle, \quad \mathbf{P}^{(p)} = d\psi\langle \mathbf{w}' * \mathbf{w}'\rangle$$

The dynamic equations for the fluid phase retain their form (2.2) if we set

$$\mathbf{q} = -\frac{1}{\langle\rho\rangle}\int \psi(\lambda)\langle\rho'\mathbf{v}'\rangle\, d\lambda, \quad \sum_{j=1}^{N} \frac{\langle\rho\rangle_j}{\langle\rho\rangle}\langle\rho'\mathbf{e}^{(j)\prime}\rangle = \frac{1}{\langle\rho\rangle}\int \psi(\lambda)\langle\rho'\mathbf{e}'\rangle\, d\lambda$$

$$\sum_{j=1}^{N} \frac{\langle\rho\rangle_j}{\langle\rho\rangle}\langle \mathbf{v}^{(j)\prime} * \mathbf{v}^{(j)\prime}\rangle = \frac{1}{\langle\rho\rangle}\int \psi(\lambda)\langle \mathbf{v}' * \mathbf{v}'\rangle\, d\lambda \tag{5.5}$$

$$\sum_{j=1}^{N} \frac{\langle\rho\rangle_j}{\sigma_j}\langle \mathbf{F}_i^{(j)}\rangle = \int \frac{\psi(\lambda)}{\sigma(\lambda)}\langle \mathbf{F}_i\rangle\, d\lambda$$

Relations (5.2) readily yield the new forms of Eqs. (4.8) and (4.9). Thus, in the case of suspensions with a continuous particle distribution we have a total of twelve equations (eight equations of conservation of mass and momentum of the phases, three transfer equations, and relation (5.1)) for twelve unknowns ($\langle p \rangle$, $\langle \rho \rangle$, the three quantities θ_i, $\psi(\lambda)$, and six velocities). This is one more equation than in the case of monodisperse suspension. However, the system of equations for polydisperse suspension is much more complex than that for a monodisperse suspension, since the equations themselves are integrodifferential.

In conclusion we note that there is generally a size dispersion of particles of equal density; however, in certain applications (ore separation in streams, separation in a pseudoliquefied layer, etc.) it is necessary to consider suspensions dispersed not only over size, but over particle density as well.

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Translated by A. Y.

DIFFUSION OF A VORTEX AND CONSERVATION OF MOMENT OF MOMENTUM IN DYNAMICS OF NONPOLAR FLUIDS

PMM Vol. 34, №2, 1970, pp. 318-323

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(Received December 3, 1969)

We show that the law of conservation of angular momentum in a flow of an incompressible Stokes fluid can, in a particular case, be reduced to the equation of vortex diffusion. We perform the analysis using two different representations, the Eulerian and the Lagrangian, of the kinetic moment of a fluid particle. We discuss the relevant concepts of the moments of inertia and give an equation for the rate of change of the Lagrangian moment of inertia of a fluid particle.

For the classical (nonpolar) media the law of conservation of the angular momentum can only lead to the condition of symmetry of the stress tensor [1], and nontrivial results can be expected only for the media with microstructure [2]. However when we consider the volumes whose characteristic dimensions are comparable with the scale of the velocity gradient field, then the balance of the angular momentum will necessarily include the kinetic moment and the mean vortical motion. Moreover it appears, that in the case of a nonpolar (e. g. Stokes') fluid, the first terms of the Taylor expansion of the kinetic moment of a particle which are not identically equal to zero, are defined by a vortex motion. We shall show that the kinetic moment of the elementary (from the point of